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The dynamical effect of inertial waves on the gyroscopic motion of a body containing several eccentrically located liquid-filled cylinders

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A theoretical analysis is made of the effect described in the title. It is shown that both (i) the eigenfrequencies of the inertial waves and (ii) the moment produced by the inertial wave pressure distribution are independent of the location of the off-axis cylinders. Hence, one can use Stewartson's tables computed for a centrally located cylinder to calculate the frequencies and residues for an eccentrically located cylinder.

1. Introduction

There exists, at present, no theoretical analysis for the dynamical effect of inertial waves on the gyroscopic motion of a body containing one or more liquid-filled off-axis cylinders as shown in figure 1. There, six liquid-filled cylinders are arranged concentrically around a seventh cylinder which is centred on the axis of the body.† To determine the effect of the inertial waves on the central cylinder, one can immediately employ the Stewartson tables and formulae (Stewartson 1959). However, it would seem obvious that such formulae and tables would not be applicable to determining the effect of inertial waves in the off-axis cylinders. Hence, an analysis of the effect of these waves when they occur in eccentrically located cylinders would be useful, and is herewith presented.

2. The fluid-dynamical equations and implications for stability

2.1. *The equations*

Following Stewartson, we assume (figure 1) that the X' and Y' axes are rotating uniformly about the Z' axis with angular speed Ω, and that the liquid is initially rotating as a rigid body with the same angular speed. Then, the velocity $\mathbf{V}'$ at some point in the liquid in *one* of the cylinders (see figure 1) is given by

$$\mathbf{V}' = \Omega\mathbf{k}\times(\boldsymbol{\xi}+\mathbf{R}). \tag{1}$$

Adopting body-fixed X, Y and Z axes, we let the perturbation of the above motion be such that the angular velocity vector of these body-fixed moving axes is subsequently given by

$$\boldsymbol{\omega} = (\omega_X, \omega_Y, \Omega) = \boldsymbol{\omega}' + (0,0,\Omega), \tag{2}$$

† Such an arrangement of cylinders is currently used in a liquid-filled artillery shell.

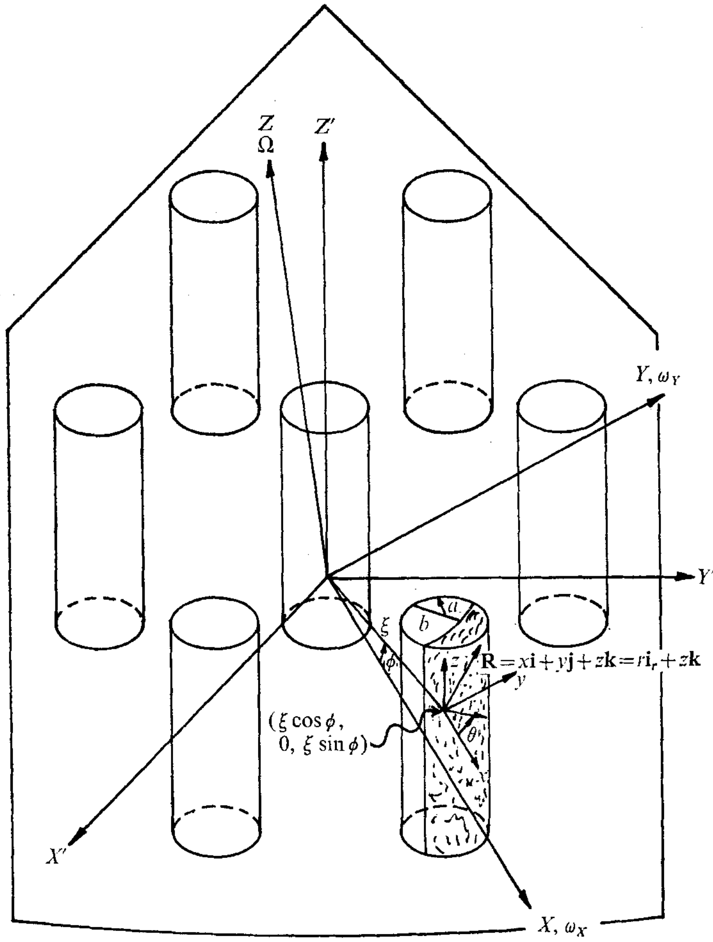

FIGURE 1. Body containing off-axis cylinders and a body-fixed co-ordinate system for a typical off-axis cylinder.

where $\boldsymbol{\omega}' = (\omega_X, \omega_Y, 0)$ is the angular velocity perturbation and is assumed small in the sense that squares and products will be neglected. Then, that point in the liquid which formerly had velocity $\mathbf{V}'$ now has velocity

$$\mathbf{V} = \Omega\mathbf{k} \times (\boldsymbol{\xi} + \mathbf{R}) + \mathbf{q}, \tag{3}$$

where $\mathbf{q}$ the velocity with respect to the original axes, i.e. the X', Y' and Z' axes characterized by the angular velocity vector $(0, 0, \Omega)$, is also assumed to be small in the same sense as $\boldsymbol{\omega}'$.

We first consider the basic unperturbed state, for which the velocity as given by (1) satisfies the Euler equation

$$\Omega^2\mathbf{k} \times [\mathbf{k} \times (\boldsymbol{\xi} + \mathbf{R})] = \mathbf{g} - \nabla P_s/\rho, \tag{4}$$

where P_s is the pressure in the uniformly rotating unperturbed liquid of density ρ. Let us determine the shape of the free surface in the eccentric cylinder before the perturbation. Following Stewartson, we assume that $\Omega^2a^2 \gg gc$. Hence, we can neglect the gravity term in (4) and since (see figure 1) $\mathbf{R} = r\mathbf{i}_r + z\mathbf{k}$,

$$\Omega^2\mathbf{k} \times [\mathbf{k} \times (\boldsymbol{\xi} + \mathbf{R})] = -\Omega^2 r\mathbf{i}_r - \Omega^2[\xi \cos\phi \cos\theta \mathbf{i}_r - \xi \cos\phi \sin\theta \mathbf{i}_\theta].$$

We can, then, rewrite (4) as

$$-\Omega^2[\xi \cos\phi \cos\theta \mathbf{i}_r - \xi \cos\phi \sin\theta \mathbf{i}_\theta] = -\nabla(P_s/\rho - \tfrac{1}{2}\Omega^2 r^2). \tag{5}$$

From the integration of (5) we have

$$P_s/\rho - \tfrac{1}{2}\Omega^2 r^2 = \Omega^2 r\, \xi \cos\phi \cos\theta + C_1, \tag{6}$$

where C_1 is a constant of integration whose determination now follows. Let $r = b-a$, $\theta = 0$ and $z = 0$ (figure 1) be the co-ordinates of some point on the free surface. Then, since $P_s = P_0$, say, there, we have

$$C_1 = P_0/\rho - \tfrac{1}{2}\Omega^2(b-a)^2 - \Omega^2(b-a)\,\xi \cos\phi. \tag{7}$$

Hence, we can rewrite (6) as

$$P_s/\rho - \tfrac{1}{2}\Omega^2 r^2 = \Omega^2 r\, \xi \cos\phi \cos\theta + P_0/\rho - \Omega^2(b-a)\,[\tfrac{1}{2}(b-a) + \xi \cos\phi] \tag{8}$$

or

$$P_s = P_0 + \rho(\tfrac{1}{2}\Omega^2)\,[r^2 - (b-a)^2] + \rho\Omega^2\xi \cos\phi[r\cos\theta - (b-a)]. \tag{9}$$

This is the expression for the pressure at any point *in* the liquid before the perturbation, and from it we can determine the steady-state free-surface configuration by setting $P_s = P_0$. Hence, from (8) we have

$$(X + \xi\cos\phi)^2 + Y^2 = (b-a)^2 + 2(b-a)\,\xi\cos\phi + \xi^2\cos^2\phi, \tag{10}$$

which, as anticipated, is a cylindrical surface of radius

$$[(b-a)^2 + 2(b-a)\,\xi\cos\phi + \xi^2\cos^2\phi]^{\frac{1}{2}}$$

with centre at $X = -\xi\cos\phi$, $Y = 0$.

We now consider the perturbed state, for which the velocity at some point in the liquid is given by (3). The linearized Euler equation now reads

$$(\partial\mathbf{q}/\partial t + 2\Omega\mathbf{k} \times \mathbf{q} - \Omega^2[\xi\cos\phi\cos\theta\mathbf{i}_r - \xi\cos\phi\sin\theta\mathbf{i}_\theta] = -\nabla[P/\rho - \tfrac{1}{2}\Omega^2 r^2], \tag{11}$$

where P is now the pressure in the perturbed liquid. Subtracting (5) from (11) we have

$$(\partial\mathbf{q}/\partial t) + 2\Omega\mathbf{k} \times \mathbf{q} = -\nabla(P - P_s)/\rho \equiv -\nabla P'/\rho, \tag{12}$$

where P', since it is small, may be termed a perturbation pressure and satisfies

$$P' = P - P_0 - \rho(\tfrac{1}{2}\Omega^2)\,[r^2 - (b-a)^2] - \rho\Omega^2\xi\cos\phi[r\cos\theta - (b-a)]. \tag{13}$$

By taking the curl of (12), then the time derivative and then the dot product with $2\Omega\mathbf{k}$, and using the result in conjunction with the divergence of (12), we get

$$\partial^2(\nabla^2 P')/\partial t^2 + (2\Omega)^2(\mathbf{k}.\nabla)^2 P' = 0, \tag{14}$$

a kind of 'wave' equation for the pressure perturbation. Following Stewartson, we seek normal-mode solutions of this equation. Hence, setting

$$P' = P_1(r, \theta, z)\, e^{st},$$

we have

$$\nabla^2 P_1 + (2\Omega/s)^2\,(\mathbf{k}.\nabla)^2 P_1 = 0, \tag{15}$$

a partial differential equation to be solved subject to appropriate boundary conditions.

It can be shown that the failure of the above-determined free surface to coincide with any of the generatrices of the offset cylinder generates a boundary condition that makes the problem mathematically intractable, Hence, we consider only the completely filled offset cylinder, for which $b = 0$ (see figure 1). Mathematically tractable boundary conditions then follow easily from the fact that, on the solid surfaces,

$$\mathbf{q}.\mathbf{n} = \boldsymbol{\omega}' \times (\boldsymbol{\xi} + n\mathbf{R}_s).\mathbf{n}, \tag{16}$$

where $\mathbf{R}_s$ refers to some point on the solid surface and $\mathbf{n}$ is the outward-directed normal. This follows from the fact that $\mathbf{q}$ is the velocity of the liquid with respect to axes having the angular velocity $(0, 0, \Omega)$, and $\boldsymbol{\omega}' \times (\boldsymbol{\xi} + \mathbf{R}_s)$ is the velocity of the corresponding point of the solid surface with respect to the same axes.

Since the governing equation (15) is in terms of P_1, we need to express $\mathbf{q}$ in terms of P_1. To do so, we set $\mathbf{q} = \mathbf{q}(r, \theta, z)\, e^{st}$ and, as before, $P' = P_1(r, \theta, z)\, e^{st}$ in (12). Then, taking the dot product and the cross product of $(2\Omega/s)\,\mathbf{k}$ with (12), respectively, and using the two results in conjunction with (12), we can show that

$$\mathbf{q} = (2\Omega/s)\,[(\mathbf{k}\times) - (s/2\Omega) - \{\mathbf{k}(2\Omega/s)\,\mathbf{k}.\}]\,\nabla P_1/\rho s[1 + (2\Omega/s)^2]. \tag{17}$$

Hence, the boundary conditions on the solid surfaces are

$$\frac{1}{\rho}\frac{\partial P_1}{\partial z}\bigg|_{z=\pm c} = -sr\,(\omega_X \sin\theta - \omega_Y \cos\theta) + s\xi\,(\cos\phi)\,\omega_Y, \tag{18}$$

$$\frac{1}{\rho}\left[\frac{\partial P_1}{\partial r} + (2\Omega/sr)\frac{\partial P_1}{\partial \theta}\right]_{r=a} = -s[z + \xi\sin\phi]\,[1 + (2\Omega/s)^2]\,(\omega_Y \cos\theta - \omega_X \sin\theta). \tag{19}$$

In these expressions, we let $P_2 = P_1 + sz\xi(\cos\phi)\,\omega_Y$. Then we can rewrite them as

$$\frac{1}{\rho}\frac{\partial P_2}{\partial z}\bigg|_{z=\pm c} = -sr\,(\omega_X \sin\theta - \omega_Y \cos\theta), \tag{20}$$

$$\frac{1}{\rho}\left[\frac{\partial P_2}{\partial r} + (2\Omega/sr)\frac{\partial P_2}{\partial \theta}\right]_{r=a} = -s[z + \xi\sin\phi]\,[1 + (2\Omega/s)^2]\,[\omega_Y \cos\theta - \omega_X \sin\theta]. \tag{21}$$

Finally, from (17) and the fact that the liquid is incompressible, we have

$$\nabla.\mathbf{q} = 0 = \partial^2 P_2/\partial r^2 + (1/r)\,(\partial P_2/\partial r) + (1/r^2)\,\partial^2 P_2/\partial \theta^2 + [1 + (2\Omega/s)^2]\,\partial^2 P_2/\partial z^2. \tag{22}$$

2.2. *Implications for stability*

For a non-eccentrically located cylinder, i.e. one whose axis lies along the Z axis and for which $\xi \sin\phi = 0$, the boundary conditions are, of course, the same as Stewartson's expressions (2.8), (2.10) and (2.4). For the off-axis cylinder, for which $\xi\sin\phi \neq 0$, calculations of the moment associated with the liquid's natural frequencies reveal that the $\xi\sin\phi$ term makes no contribution; i.e. the expression with which it is functionally involved integrates to zero. Hence, the 'effective' terms on the right-hand sides of (20) and (21) are identical to Stewartson's expressions (2.8) and (2.10), and the liquid in each off-axis cylinder has an effect indistinguishable from that in the centred cylinder. One can, then, use Stewartson's tables and formulae for the eccentric cylinders as well as for the central one (the net residue or moment will be the sum of those for each cylinder).

Physically, these conclusions are reasonable, for the moments due to the liquid oscillations are couples, and the moments of couples are independent of their location; also the liquid frequencies are natural frequencies, the values of which are independent of the nature of the disturbance that generated them.

3. Summary

The dynamical effect of inertial waves on the eccentric cylinders is as though each cylinder were located on the axis of the shell, each having the same eigenfrequencies. Hence, one can use Stewartson's tables and formulae, computed for a centrally located cylinder, to determine the eigenfrequencies and moments (residues) for the liquid in each off-axis cylinder. However, the net moment (residue) is the sum of the moments for each cylinder. Symmetry considerations for the equations of motion of the system would demand, of course, that there be an even number of cylinders surrounding the central one.

REFERENCE

Stewartson, K. 1959 The stability of a spinning top containing a liquid. *J. Fluid Mech.* **5**, 577–592.